MATEMATİK BÖLÜMÜ

HOPF BIFURCATION AND STABILITY ANALYSIS FOR CONTINUOUS NEURAL NETWORK MODEL WITH DISTRIBUTED DELAY

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ABSTRACT

Dynamical systems theory is a branch of mathematics and physics that studies the evolution of systems over time. These systems can be as simple as the motion of a pendulum or as complex as the behavior of the weather or the dynamics of a population. The key idea is to understand how a system changes and evolves, often in response to its initial conditions and external influences. There are various studies that study neural networks through the lens of dynamical systems. Modeling neural networks as dynamical system enhances our knowledge about one of the most important concept in neuroscience; neurons. In this thesis, we provide fundamental insights into dynamical systems, exploring notable examples and demonstrating the process of linearizing a system around an equilibrium point. Additionally, we delve into bifurcations in both one and two dimensions. Our examination extends to a neural network model, where we illustrate the occurrence of Hopf bifurcation under specific circumstances

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Dynamical systems are equations or systems of equations that describe a system changing over time. A dynamical system has a state vector that indicates the current state and a function that determines what will happen in the future, given the current state, within the system. In a more mathematical notation, we denote the current state as a vector $\mathbf{x}$ and the function $\mathbf{x}$ that takes the current state as an input. At the origin, we can consider two types of dynamical systems: discrete-time systems and continuous-time systems.

For continuous time dynamical system, we denote time by t, and the following equations specify the system:

$$\mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{x}(k+1) = f(\mathbf{x}(k)),$$

Example*(Population Model)*
Population models are one examples of dynamical systems. Now we will give the simplest population model ever. In this model we assume population growth is proportional to population itself. We denote the population size at the time k as $\mathbf{x}_k$, then we have $x_{k+1} = f(x_k) = x_k + ax_k = cx_n$ where a > 0 is a constant.
For this model we can write the closed formula for the population at time k:

$$x_k = f^k(x) = c^k x_0$$

for values c > 1, the population grows exponentially and for c < 1, population shrinks exponentially.

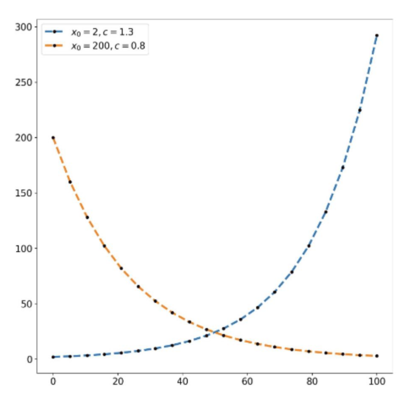

Figure 1.1: Simple Population Model; $x_k = c^k x_0$

Example*(The Lorentz System)*
The system of differential equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\beta - z) - y$$
$$\dot{z} = xy - \rho z$$

is called Lorentz system where σ, β, ρ are constants.It stands out as one of the well-known dynamical systems. Lorentz system lives in $\mathbb{R}^{3}$. It is an example of what one calls a strange attractor.

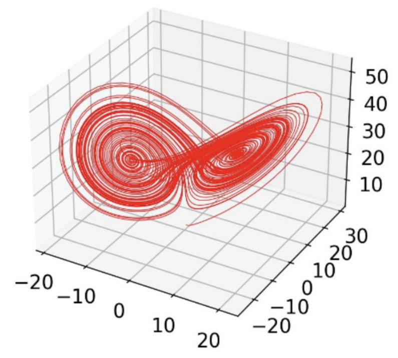

Figure 1.2: Lorentz system

$$\dot{x} = a_{11}(\tau)x + a_{12}(\tau)y + f_1(x, y, \tau)$$
$$\dot{y} = a_{21}(\tau)x + a_{22}(\tau)y + g_1(x, y, \tau).$$

Theorem*(Hopf Bifurcation Theorem)*
Let f_1 and g_1 in the system above possess continuous third-order partial derivatives with respect to both x and y. Assume that the origin serves as an equilibrium point for (8) and that the Jacobian matrix J(τ) defined above remains valid for all sufficiently small |τ |. Furthermore, suppose that the eigenvalues of the matrix J(τ) are given by α(τ) ± iβ(τ), where α(0) = 0 and β(0) / = 0 such that the eigenvalues cross the imaginary axis with nonzero speed, i.e.

$$\frac{\alpha}{\tau}|_{\tau=0} \neq 0$$

Then in any open set U containing the origin in R2 and for any τ0 > 0, there exists a value $\bar{\tau}, |\bar{\tau}| < \tau_0$ such that the system of differential equations has a periodic solution for τ = τ̄ in U.

Neurons as Dynamical Systems

One can consider neurons as one of the most important concept of neuroscience. Neurons, the fundamental building blocks of the nervous system, play a crucial role in transmitting information within the intricate network of the brain. These remarkable cells are not only responsible for receiving and sending signals but also exhibit dynamic behaviors that can be studied through the lens of dynamical systems theory.

In that sense, neurons can be viewed as dynamic entities with complex interactions that give rise to the emergent properties of neural networks. A neuron's activity is not a static process but rather a dynamic one, characterized by the flow of electrical and chemical signals across its membrane. The integration of inputs, the generation of action potentials, and the release of neurotransmitters contribute to the dynamic nature of neuronal function.

Dynamical systems theory provides a framework for understanding and modeling the neuronal activity. Studying neurons as dynamical systems provides valuable insights into the fundamental principles governing neural function. It enables researchers to explore how dynamic interactions between neurons contribute to cognitive processes and how perturbations in these dynamics may underlie neurological disorders. Overall, the application of dynamical systems theory to the study of neurons enhances our understanding of brain.

In this thesis we examined the below system containing both discrete and distributed delays:

$$\dot{x}_1 = -\mu x_1(t) + a_{11} f_{11}\Big(\int_{-\infty}^{t} F(t-s) x_1(s-\tau) ds\Big) + a_{12} f_{12}(x_2(t-\tau)))$$
$$\dot{x}_1 = -\mu x_2(t) + a_{21} f_{21}(x_1(t-\tau)) + a_{22} f_{22}\Big(\int_{t}^{-\infty} F(t-s) x_2(s-\tau ds)\Big)$$

where x_i's are the state variables for two neurons τ is synaptic delay. $\mu > 0$. $f_{ij} : \mathbb{R} \to \mathbb{R}(f_{ij}(0) = 0, f_{ij} \in \mathbf{C}^r(\mathbb{R}), r \geq 4)$
The weight function F (•) is non-negative bounded function that stands for the impact of past states on the current dynamics. We want to reduce the number of parameter. Hence we introduce the following parameters to the system:

$$x_3 = \int_{-\infty}^{t} F(t-s) x_1(s-\tau) ds$$
$$x_4 = \int_{-\infty}^{t} F(t-s) x_2(s-\tau) ds$$

Thus, our new system will be as follows:

$$\dot{x}_1 = -\mu x_1(t) + a_{11} f_{11}\Big(\int_{-\infty}^{t} F(t-s) x_1(s-\tau) ds\Big) + a_{12} f_{12}(x_2(t-\tau)))$$
$$\dot{x}_1 = -\mu x_2(t) + a_{21} f_{21}(x_1(t-\tau)) + a_{22} f_{22}\Big(\int_{t}^{-\infty} F(t-s) x_2(s-\tau ds)\Big)$$
$$\dot{x_3} = \mu x_1(t-\tau) - \mu x_3(t)$$
$$\dot{x_4} = \mu x_2(t-\tau) - \mu x_4(t)$$

Then we linearize the system around the equilibrium point to use linear stability analysis and calculate the system's Jacobian matrix and following its characteristic polynomial:

$$\lambda^4 + 4\lambda^3\mu + 6\lambda^2\mu^2 + \lambda(4\mu^3 - 4b_{22}) + \mu^4 - b_{22}\mu$$
$$+ e^{-2\lambda\tau}\big(b_{12}b_{21}\lambda^2 + 2b_{12}b_{21}\mu\lambda - (b_{11}b_{22} - b_{12}b_{21})\mu^2 + e^{\lambda\tau}(b_{11}\mu\lambda^2 + 2\mu b_{11}\lambda + \mu^3 b_{11})\big) = 0$$

Then we define new coefficients to make the equation friendlier

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$
$$+ e^{-2\lambda\tau}\big(b_2\lambda^2 + b_1\lambda + b_0 + e^{\lambda\tau}(c_1\lambda^2 + 2c_1\lambda + c_0)\big) = 0$$

Theorem. If we take τ = 0 for the system (4.10) the equilibrium point is asymptotically stable.
Proof. To prove the theorem we substitute τ = 0 and we find the conditions that coefficients of the polynomial

$$\lambda^4 + a_3\lambda^3 + (a_2 + b_2 + c_1)\lambda^2 + (a_1 + b_1 + 2c_1)\lambda + a_0 + b_0 + c_0 = 0$$

holds for Routh-Hurwitz Criterion. Consequently, we showed that when τ = 0 the roots have negative or negative real part. Thus, it is asymptotically stable.
Lemma. For the friendlier polynomial, the transcendental equation have imaginary roots.
Proof. We take w>0 ∈ R and τ =τ*, λ=iw. Then

$$w^4 - a_3 i w^3 - a_2 w^2 + a_1 i w + a_0 +$$
$$(\cos 2w\tau - i \sin 2w\tau)(-b_2 w^2 + b_1 i w + b_0 + (\cos w\tau + i \sin w\tau)(-c_1 w^2 + 2c_1 i w + c_0)) = 0$$

We split the above equation into real and imaginary parts. Then we solve the two equation for τ. By solving the equation system, we showed there exist τ_0 and if τ > τ0 then the equilibrium point is unstable.
Lemma. Let τ = τk and k = 0, 1, 2, ... then we have the following

$$Re\frac{d\lambda(\tau_k)}{d\tau} \neq 0$$

where

$$\lambda(\tau) = -(\lambda+\mu)^4 + (\lambda+\mu)b_{22} + e^{-2\lambda\tau}\big(\mu^2 b_{11}b_{22} - (\lambda+\mu)^2(b_{21}b_{12})\big)$$
$$-e^{-\lambda\tau}\big((b_{11}\mu)(\lambda+\mu)^2\big) = 0$$

MAIN RESULT

Theorem. For the system we examined the following conditions are satisfy**:**

- If τ ∈ [0,τ_0) then the equilibrium point is asymptotically stable.
- If τ > τ_0 then the equilibrium point is unstable.
- Hopf bifurcation occurs at the equilibrium point if τ = τ_s, (s = 0, 1, 2, ...)

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